Dondi Ellis

1 First Question

To the first question, a mathematician would say that the polynomial $f(x) = 0$ is indeed a polynomial with an emphatic yes. But, one needs a good working definition of what is meant by a polynomial. One probably first encounters a somewhat rigorous definition of the term polynomial in abstract algebra. In abstract algebra we talk about adjoining a Ring R with an indeterminate x, written $R[x]$. This forms another ring, and by definition the new ring $R[x]$ consists of all formal finite sums of elements $a_0 + a_1x + ... + a_nx^n$ where each $a_i \in R$. By definition, a Ring R is a set with an operation $+$ called addition and an operation $\cdot$ called multiplication such that R is an abelian group under addition and such that R is closed under multiplication. This requires me to define what a group is.

Definition. A group $(G, *)$ is by definition a set G together with an operation $*$ such that for all $a, b, c \in G$ the following holds:

1. $a * b \in G$. (closure)
2. $a * (b * c) = (a * b) * c$. (associativity)
3. There exists an element $1 \in G$ called the identity element such that for all $g \in G$ we have that $g * 1 = 1 * g = g$. (identity)
4. For each element $a \in G$ there exists an element b such that $a * b = b * a = 1$. (inverses)

By abuse of language one writes G for the group $(G, *)$ and says that G is a group under the operation of $*$. If G satisfies the further commutativity property that for all $a, b \in G$ we have that $a * b = b * a$ then we say that G is abelian.

This all is a bit abstract, but I assume that by a polynomial you mean that the coefficients are either real or complex numbers. Lets assume that the numbers are complex. We denote the set of all complex numbers by $\mathbb{C}$. $\mathbb{C}$ forms a ring under the obvious operations of a addition and multiplication. The ring $\mathbb{C}[x]$ is the polynomial ring over $\mathbb{C}$ and its additive identity element is the polynomial 0.

In terms of linear algebra we have that the set of all polynomials in the variable x over the field of complex numbers $\mathbb{C}$ form a Vector Space.

Definition. A vector space over a field F is by definition a set V along with operations $\cdot$ and $+$ called multiplication and addition respectively such that V forms an abelian group under $+$ and such that for all $\lambda \in F$ and $v \in V$ we have that $\lambda \cdot v \in V$.

I guess I haven't really defined what a field is. A field F is a very nice example of a ring. If 0 denotes the identity element of the operation $+$ in F then F is a field if it is an abelian group under $+$ and a group under $\cdot$ with 0 taken out. For example, $\mathbb{C}$ forms a field. One sees that all finite formal sums of elements in $\mathbb{C}[x]$ form a vector space with additive identity the element $0 \in \mathbb{C}[x]$.

Basically there is no reason that one would want to exclude the identity element 0 from the set of polynomials. In a lot of ways 0 is one of the most important polynomials in the ring, for it is the unique additive identity. Moreover, if you say 0 is not a polynomial, then you are saying that it is possible to add two polynomials together and get something which is not a polynomial, and thus the set of polynomials is not closed under addition.

The short answer is that no research mathematician, or college textbook for that matter, would ever define 0 to not be a polynomial. It would make about zero sense to define it this way.

2 Second Question

For the second question, my answer would be yes in the same sense that $\arctan(x), \arcsin(x), ect.$ are functions. The inverse of the $\tan, \sin, ect$ functions are only 1-1 if you restrict the domain appropriately. So essentially the standard definition of say $\arctan(x)$ is arbitrary. We could have just as easily defined to on any other connected open interval of length π. We choose the interval $(-\pi/2, \pi/2)$ simply because it goes through the origin and is an interval on which the function is continuous. We could have just as easily chose the interval $(-\pi/2 + n\pi, \pi/2 + n\pi)$ for any integer n.

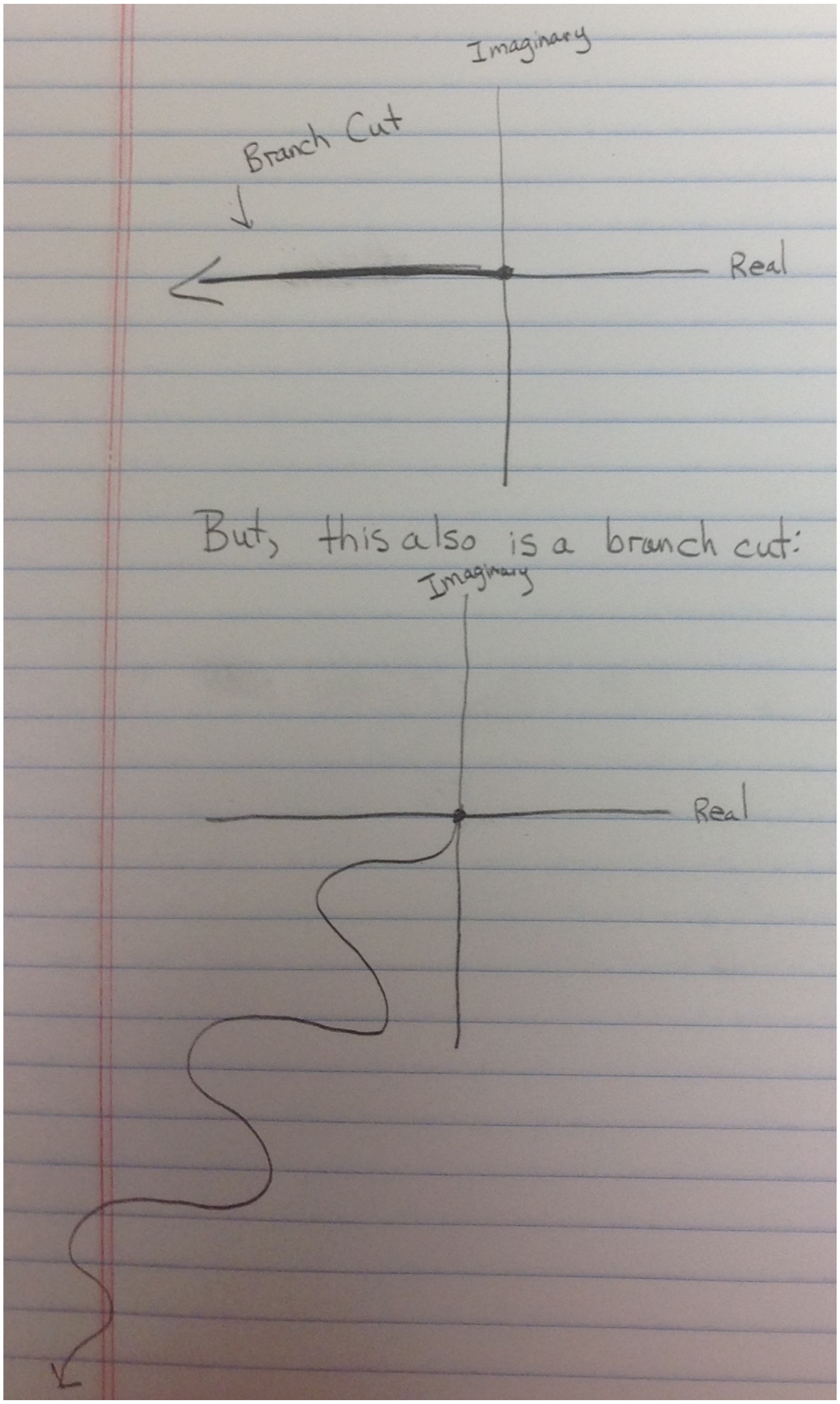
This question belongs to the field of complex analysis, and is actually quite involved. Most likely an advanced junior or senior undergraduate in some sort of pure math program has seen this, but I would put the question at about that level. With that being said its not a particularly difficult question, it just requires some background. So in one-variable calculus we deal with differentiation of real valued functions. Then in multivariable calculus one learns about what differentiation and integration mean in the context of several variables. A first year Complex Analysis course is sorta the complex number analogue of Calculus 1. One defines the derivative in complex analysis just as one does in calculus one. That is, $f(z)$ is said to be differentiable at z_0 iff:

$$\lim_{h \to 0} \frac{f(z_0 + h) - f(z_0)}{h} \text{ exists.}$$

The difference is that h is allowed to be a complex number and so h can approach z_0 from any angle, which appears to be somewhat hard to check. It turns out that this limit exists iff the function $f(z)$ satisfies what are know as the Cauchy-Riemann equations. it turns out that the natural logarithm function turns out to have some problems that are both somewhat annoying and helpful. We denote the natural logarithm function by Log(z) in complex analysis. Talking about the logarithm of complex numbers does not exactly make sense. So first we define the complex continuation of the exponential function e^z as follows. If θ is a real number, we define $e^{i\theta} = \cos\theta + i\sin\theta$. One can check that $e^{i(\theta_1+\theta_2)} = \cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2) = (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) = e^{i\theta_1}\cdot e^{i\theta_2}$ and so all of the normal exponential rules apply. So define e^z for $z = x + iy \in \mathbb{C}$ by $e^{x+iy} = e^x \cdot e^{iy}$ and we see that e^z is actually an entire function. This means that it is complex differentiable on all of $\mathbb{C}$. One fact of of complex differentiable functions is that a complex differentiable function is infinitely differentiable. This is unlike the real case. Consider for example $f(x) = x^{3/2}$. It turns out that this is the unique way to extend e^x to all complex numbers in a continuous manner. This follows immediately from the Identity Principle of Complex Analysis.

Before one can make sense of i^x, one needs a knowledge of the above. What is the domain of our function i^x? It should be an open subset of $\mathbb{C}$. Notice that $i = \cos(\pi/4 + 2n\pi) + i\sin(\pi/4 + 2n\pi) = e^{i(\pi/4+2n\pi)}$ for any integer n. So, we must have that $i^x = e^{i(\pi/4+2n\pi)x}$. This doesn't really make sense since our answer depends on our choice of n, and in fact if x is say the square root of 2 then we have that going through all n gives us a dense subset of the unit circle.

Ok, so lets take a step back. Notice that in the real case we can define, for example, the function a^b as $e^{b\ln a}$. The correct thing to do is to define a^b to be $e^{b\ln a}$ for $a, b \in \mathbb{C}$. So I interpret the function $f(z) = i^z$ to mean $f(z) = e^{zLog(i)}$. But this requires me to define the complex logarithm function Log(z). This should in some sense be an inverse to e^z. It turns out that the correct definition is to define $\text{Log}(z) = \ln|z| + iArg(z)$. Here $Arg(z)$ is the argument of z. Notice that this number is only unique up to an integer multiple of $2\pi i$. It turns out that we have to make a choice of what is called a branch cut, and different choices will give us different functions! There is no obvious choice, and in fact we don't make a choice, and we have to specify each time what branch cut we are using. A brach cut must include the orgin, and it must be some curve not crossing itself and must make it such that deleting this curve makes the the remaining points on the complex plane simply connected. Simply connected basically means that any closed curves in the set don't enclose holes. One example of a branch cut would be to delete $\{x + i\cdot 0 | x \leq 0\} \subset \mathbb{C}$. This gives us a function Log (z) uniquely defined after a choice of branch cut. In this manner we can define $f(z) = i^z = e^{zLog(i)} = e^{z(\ln|i| + iArg(z))} = e^{izArg(z)}$.



One thing to notice is that your question would have been no different had you asked me how to extend the definition of 2^z to complex numbers. This is actually an equally frustrating question.

It turns out that these concepts are extremely useful. For instance, you can use these kinds of functions to solve the following indefinite integral. There is no way of doing this without the aid of complex analysis, although it appear to be a problem from one variable calculus:

$$\int_0^\infty \frac{x^t}{(x+1)(x+2)}dx, \text{where } t \in (0,1).$$